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1. Introduction

Let $X_1, X_2, \cdots, X_n, \cdots$ be mutually independent and identically distributed random variables with a common cdf $F(t) = P\{X \leq t\}$ such that $E[X^+] = \int_{R_+} tdF(t) < \infty$, where $R = (-\infty, \infty), R_+ = [0, \infty)$. A positive observation cost $c(\in R_{++} = (0, \infty))$ is incurred to the observation of each $X_n, n \geq 1$. If the observation process is stopped after X_n is observed, a reward $X_n - nc$ is received.

The optimal stopping time N is necessarily of the form; to stop at $N = \min\{n \mid X_n \in S\}$ for some stopping set $S \subset R$, and S is stationary and of a control-limit-type $\{X \geq x\}$ or $\{X > x\}$ for some $x \in R$, where x is called a stopping level. For this, we define that a stopping level x (or $x - 0$) means a stopping set $\{X > x\}$ (or $\{X \geq x\}$) respectively.

For any stopping level x and for any cdf F, we define an expected reward $\phi(x, F) = E[X_N - cN]$ of the stopping problem by

$$\phi(x, F) = \frac{\int_{(x,\infty)} tdF(t) - c}{\bar{F}(x)} = x + \frac{\int_{(x,\infty)}(t - x)dF(t) - c}{\bar{F}(x)}, \tag{1.1}$$

where $\bar{F}(x) = 1 - F(x)$. Note that $\bar{F}(x) \to 0$ and $\phi \to -\infty$ as $x \to \infty$ and that $\bar{F}(x) \to 1$ and $\phi \to \mu_F - c$ as $x \to -\infty$ where $\mu_F = E[X] = \int_R tdF(t)$.

By the assumption $E[X^+] < \infty$, define $T_F(x)$,

$$T_F(x) = \int_x^\infty (t - x)dF(t) = \int_x^\infty \bar{F}(t)dt . \tag{1.2}$$

Lemma 1. $T_F(x)$ is continuous, non-negative, convex and non-increasing function of x. It satisfies that $T_F(x) \geq (\mu_F - x)^+$ for any $x \in R$ and that $T_F(x) \to +\infty$ as $x \to -\infty$ and $T_F(x) \to 0$ as $x \to +\infty$. T_F has a derivative a.e.. Moreover, if $T_F(x)$ is positive at any point x, it is strictly decreasing at x.

Now, redefining the expected reward $\phi(x, F)$ by (1.1′) for any stopping level x and for any cdf F, we will have the optimal expected reward $\phi^o(F)$ for any cdf F.

$$\phi(x, F) = x + \frac{T_F(x) - c}{\bar{F}(x)} . \tag{1.1′}$$

(1.3)
$$\phi^o(F) \overset{\text{def}}{=} \sup_{x \in R} \phi(x, F) .$$

(1.4)
$$\frac{d\phi(x, F)}{dF(x)} = \frac{T_F(x) - c}{\bar{F}^2(x)} .$$

The right hand side of (1.4) changes the sign from $+$ to $-$ at most one time as x goes from $-\infty$ to $+\infty$. From Lemma 1, the equation $T_F(x) = c$ for any fixed $c (c > 0)$ has a unique solution $x^o(F) \overset{\text{def}}{=} (T_F)^{-1}(c)$, so that the set of optimal stopping levels $\mathbf{x}^o(F)$ (which must contain the point $x^o(F)$) of (1.3) is given by

(1.5)
$$\mathbf{x}^o(F) = \{x \mid F(x) = F(x^o(F))\} .$$

Since the cdf F is right-continuous, this set is an interval of the form $[a, b)$.

We have the optimal expected reward $\phi^o(F)$,

(1.3′)
$$\phi^o(F) = x^o(F) = \phi(\mathbf{x}^o(F), F) ,$$

where $\phi(A, F)$ means $\phi(y, F)$ for any y in a set A.

Lemma 2. For any given cdf F, the following stopping sets or stopping levels (i) (ii) (iii) are optimal, and the optimal expected reward is given by (1.3″) ;

(i) the set $\{X > a\}$ or level a where $a = \min\{x \mid x \in \mathbf{x}^o(F)\}$,
(ii) the set $\{X \geq b\}$ or level $b - 0$ where $b = \sup\{x \mid x \in \mathbf{x}^o(F)\}$,
(iii) the set $\{X > x\}$ ($\{X \geq x\}$) or level x ($x - 0$) where $\forall x \in (a, b)$.

First, we shall derive the maximal bound ϕ^u for $\phi(x, F)$ on $R \times \mathcal{F}$

(1.6)
$$\begin{aligned} \phi^u &= \sup_{x \in R} \sup_{F \in \mathcal{F}} \phi(x, F) = \sup_{F \in \mathcal{F}} \phi^o(F) \\ &= \phi(x^o(F^u), F^u) = \phi(x^u, F^u) , \end{aligned}$$

where (x^u, F^u) is a joint maximizing point of $\phi(x, F)$.

Second, we shall consider $\phi(x, F)$ as a two-person zero-sum game in which the player 1 (gambler) decides his level x in R and the player 2 (nature) chooses her cdf F in $\mathcal{F}$, before the observation of $\{X_n; n \geq 1\}$. Then the minimax value ϕ^* and the maximin value ϕ_* on $R \times \mathcal{F}$,

(1.7)
$$\begin{aligned} \phi^* &= \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F) = \inf_{F \in \mathcal{F}} \phi^o(F) \\ &= \phi(x^o(F^*), F^*) = \phi(x^*, F^*) , \end{aligned}$$

(1.8)
$$\phi_* = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) = \phi(x_*, F_*) ,$$

and the saddle value ϕ^s, the saddle point (x^s, F^s) in $R \times \mathcal{F}$,

$$(1.9) \qquad \phi^s \;=\; \text{value}_{x\in R, F\in\mathcal{F}}\phi(x,F) \;=\; \phi(x^s, F^s)\ ,$$

will be derived for the following two classes $\mathcal{F}(\mu,\sigma^2)$ and $\mathcal{F}(\mu,\sigma^2,M)$ of cdf's.

The class $\mathcal{F}(\mu,\sigma^2,M)$ is the set of cdf's whose mean μ, variance σ^2 and domain $[\mu - M, \mu + M]$ are assumed to be known.

$$(1.10) \qquad \mathcal{F}(\mu,\sigma^2,M) \;=\; \{F \mid \int_A dF(t) = 1, \int_A t dF(t) = \mu,$$

$$\int_A t^2 dF(t) = \mu^2 + \sigma^2 \text{ where } A = [\mu - M, \mu + M]\ ,\ M \geq \sigma\}$$

The class $\mathcal{F}(\mu,\sigma^2)$ is $\mathcal{F}(\mu,\sigma^2,M)$ where M is arbitrary in R_{++}, and $\mathcal{F}(\mu)$ is $\mathcal{F}(\mu,\sigma^2)$ where σ^2 is arbitrary in R_{++}.

Let a random variable X has a mean μ with a cdf $F_\mu(t)$, then the new random variable $X-\mu$ has the mean 0 with the cdf $F_0(t) = F_\mu(t+\mu)$. The following Lemma 3 below holds immediately from the definition (1.1) of $\phi(x,F)$.

2. Some Fundamental Lemmas

Lemma 3.

$$(2.4) \qquad \phi(x, F_\mu) \;=\; \mu \;+\; \phi(x-\mu, F_0) \text{ for any } x \in R\ .$$

Therefore, we may assume without loss of generality that all the cdf's in F have the mean 0. So that, we shall analyze the stopping problem in only two classes $\mathcal{F}(0,\sigma^2)$ and $\mathcal{F}(0,\sigma^2,M)$.

Lemma 4. For cdf's F_i and non-negative numbers $\lambda_i, i = 1,2,\cdots,n$, such that $\sum_{i=1}^n \lambda_i = 1$, let $F = \sum_{i=1}^n \lambda_i F_i$. Then

$$(2.5) \qquad \phi(x,F) \;=\; \sum_{j=1}^{n} \lambda_j(x)\phi(x,F_j) \quad \text{for any } x \in R\ , \text{ where}$$

$$\lambda_j(x) \;=\; \frac{\lambda_j \bar{F}_j(x)}{\sum_{i=1}^n \lambda_i \bar{F}_i(x)}.$$

Let define G_n be a discrete cdf which has n probability masses p_i, $p_i > 0$, at n points t_i, $i = 1,2,\cdots,n$, respectively ($\sum_{i=1}^n p_i = 1$), i.e., it is represented as

$$(2.6) \qquad G_n(t) \;=\; (< t_1,\cdots,t_n >< p_1,\cdots,p_n >)\ ,$$

and $\mathcal{G}_n(\mu,\sigma^2)$ be all discrete cdf's G_n in $\mathcal{F}(\mu,\sigma^2)$. Let

$$(2.7) \qquad G_2(t;q) \;=\; (< -\frac{\sigma}{q}, \sigma q >< \frac{q^2\ ,\ 1}{1+q^2} >)$$

for any $q, 0 < q < \infty$. Then $G_2(t;q)$ is the only two-point cdf which has the mean 0 and the variance σ^2.

Lemma 5. The class $\mathcal{G}_2(0,\sigma^2)$ of two-point cdf's is represented with a parameter q, $0 < q < \infty$, as follows,

$$\mathcal{G}_2(0,\sigma^2) \ = \ \{G_2(\cdot;q) \ | \ 0 < q < \infty\} \ .$$

Let us define

$$T^u_{\mathcal{F}}(x) \ = \ \sup_{F\in\mathcal{F}} T_F(x) \ , \ T^\ell_{\mathcal{F}}(x) \ = \ \inf_{F\in\mathcal{F}} T_F(x) \ . \tag{2.8}$$

Lemma 6. Suppose $\mathcal{F} = \mathcal{F}(0)$ so that $\mu_F = 0$ for all $F \in \mathcal{F}$, then $T^u_{\mathcal{F}}(x)$ and $T^\ell_{\mathcal{F}}(x)$ have the same property as $T_F(x)$ in Lemma 1 with μ_F replaced by 0 , except that $T^\ell_{\mathcal{F}}(x)$ is not always convex.

From above Lemma 6, $T^u_{\mathcal{F}}(x)$ and $T^\ell_{\mathcal{F}}(x)$ have inverse functions $(T^u_{\mathcal{F}})^{-1}(c)$ and $(T^\ell_{\mathcal{F}})^{-1}(c)$ for all $c, c > 0$, respectively. Thus we have shown the existence of the values of ϕ^u and ϕ^*:

$$\phi^u \ = \ \sup_{F\in\mathcal{F}}\{x \mid T_F(x) = c\} \ = \ (T^u_{\mathcal{F}})^{-1}(c) \ , \tag{2.9}$$

$$\phi^* \ = \ \inf_{F\in\mathcal{F}}\{x \mid T_F(x) = c\} \ = \ (T^\ell_{\mathcal{F}})^{-1}(c) \ . \tag{2.10}$$

3. The Class $\mathcal{F}(\mu,\sigma^2)$

Proposition 3. [Feller p.151] If F is an arbitrary cdf, then

$$\Big(\int_A u(t)v(t)dF(t)\Big)^2 \leq \Big(\int_A u^2(t)dF(t)\Big)\Big(\int_A v^2(t)dF(t)\Big) \tag{3.2}$$

for any set A and any functions u, v for which the integrals on the right exist. Furthermore, the equality sign holds if and only if

$$\int_A (au(t)+bv(t))^2 dF(t) = 0 \text{ for some } a, b \in R \ . \tag{3.3}$$

Note that if u and v are linearly dependent, i.e., for some $a, b \in R$, $au(t)+bv(t) = 0$, the condition (3.3) is satisfied for all $F \in \mathcal{F}$, and that if u and v are linearly independent, the condition (3.3) is satisfied only when the cdf F is degenerated at one point in a set A.

We shall calculate ϕ^u and the maximizing point (x^u, F^u) of the problem (2.9) by Proposition 3.

$$\Big(\int_{(x,\infty)} (t-x)dF(t)\Big)^2 \leq \Big(\int_{(x,\infty)} dF(t)\Big)\Big(\int_{(x,\infty)} (t-x)^2 dF(t)\Big) \ , \tag{3.4}$$

$$(3.4') \qquad \left(\int_{(-\infty,x]}(t-x)dF(t)\right)^2 \le \left(\int_{(-\infty,x]} dF(t)\right)\left(\int_{(-\infty,x]}(t-x)^2 dF(t)\right) .$$

Then, we obtain the maximal bound ϕ^u.

$$(3.7) \qquad \phi^u = \sup_{F\in\mathcal{F}}\{x \mid T_F(x) = c\} = \frac{\sigma^2}{4c} - c .$$

Since the equality holds in two Schwartz inequalities (3.4) and (3.4′), from the remark of Proposition 3, the maximizing cdf F^u should be the two-point cdf. Then, we have

$$(3.8) \qquad F^u(t) = G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} >< \frac{\sigma^2}{\sigma^2+4c^2}, \frac{4c^2}{\sigma^2+4c^2} >) ,$$

$$(3.9) \qquad x^u \in \mathbf{x}^u = \mathbf{x}^o(F^u) = [-2c, \frac{\sigma^2}{2c}) .$$

Theorem 1. For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the maximal bound ϕ^u is $\sigma^2/4c - c$ by (3.7) and the maximizing point $(x^u, F^u) \in \mathbf{x}^u \times \{F^u\}$ is given by $F^u(t) = G_2(t;\sigma/2c)$ in (3.8) and $\mathbf{x}^u = [-2c, \sigma^2/2c)$ in (3.9).

Remark of Theorem 1. From Lemma 2, The equation (3.9) means that the player 1 may decide a stopping level x^u for some $x^u \in [-2c, \sigma^2/2c)$ or $\sigma^2/2c - 0$. If the player 1 decides any of the above stopping levels, he stops the process whenever $X_n = \sigma^2/2c$ is observed because the player 2 chooses only one cdf given by (3.8).

Second, we shall calculate the minimax value ϕ^* of (2.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$.

From Lemma 6, $T^\ell_{\mathcal{F}}(x) \ge (-x)^+$ for all $x \in R$. Then it holds that $T_{F^*}(x) = (-x)^+ \le T^\ell_{\mathcal{F}}(x)$ for $x \in (-\infty, -c]$ if a cdf F^*, which has all the mass on $[-c, \infty)$, is contained in $\mathcal{F}$. Since $T_{F^*}(x) = (-x)^+$ is strictly decreasing on $(-\infty, -c]$, we have

$$(3.10) \qquad \phi^* = \inf_{F\in\mathcal{F}}\{x \mid T_F(x) = c\} = \{x \mid T_{F^*}(x) = c\} = -c .$$

Such a class $\mathcal{F}^*$ of cdf's F^* always exists in $\mathcal{F}$ for all c, $c > 0$.

$$(3.11) \qquad \mathcal{F}^* = \{F \mid \int_{[-c,\infty)} dF(t) = 1,\ F \in \mathcal{F}\} .$$

In particular, we can find the class $\mathcal{G}_2^* = \mathcal{G}_2^*(0,\sigma^2)$ of two-point cdf's in $\mathcal{F}^*$ from Lemma 5.

$$(3.11') \qquad \mathcal{G}_2^* = \{G_2(\cdot; q) \mid q \ge \frac{\sigma}{c}\} .$$

It is easily shown that for any $F^* \in \mathcal{F}^*$ it is optimal for the player 1 to stop the process immediately. That is,

$$(3.12) \qquad \mathbf{x}^* = \mathbf{x}^o(F^*) = (-\infty, -c) \text{ for all } F^* \in \mathcal{F} .$$

Theorem 2. For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the minimax value ϕ^* is $-c$ by (3.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ is given by (3.11) and (3.12). In particular, there exists the class $\mathcal{G}_2^*$ of two-point cdf's in $\mathcal{F}^*$ by (3.11′).

Now, we shall derive the saddle value ϕ^s for $\phi(x,F)$ in $\mathcal{F} = \mathcal{F}(\mu,\sigma^2)$. We have a candidate $(\mathbf{x}^*, \mathcal{F}^*)$ for a set of saddle points $(\mathbf{x}^s, \mathcal{F}^s)$.

Theorem 3. For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the saddle value ϕ^s is $-c$ and the saddle point $(x^s, F^s) \in \mathbf{x}^s \times \mathcal{F}^s$ is given by $\mathbf{x}^s = \mathbf{x}^*, \mathcal{F}^s = \mathcal{F}^*$ and $\mathcal{G}_2^s = \mathcal{G}_2^* \subset \mathcal{F}^s$ defined in Theorem 2.

Theorem 3 says the class $\mathcal{F}(\mu,\sigma^2)$ is so rich for the player 2 that the player 1 must stop immediately. In this case, the information of the value σ^2 is useless for the player 1.

4. The Class $\mathcal{F}(\mu,\sigma^2,M)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s in the more restrictive and interesting class $\mathcal{F} = \mathcal{F}(0,\sigma^2,M)$ (see (1.10)).

Theorem 4. For a class $\mathcal{F}(0,\sigma^2,M)$ of cdf's, $\sigma < M$, the maximal bound ϕ^u and the maximizing point $(x^u, F^u) \in \mathbf{x}^u \times \mathcal{F}^u$ are as follows:
(i) When $0 \le c \le \sigma^2/2M$,

$$\phi^u = M - c(1+\frac{M^2}{\sigma^2})\ ,\ \mathbf{x}^u = [-\frac{\sigma^2}{M}, M)\ ,$$

$$F^u(t) = G_2(t;\frac{M}{\sigma}) = (< -\frac{\sigma^2}{M}, M >< \frac{M^2,\sigma^2}{\sigma^2+M^2} >)\ .$$

(ii) When $\sigma^2/2M \le c \le M/2$, the same result as Theorem 1 holds, i.e.,

$$\phi^u = \frac{\sigma^2}{4c} - c\ ,\ \mathbf{x}^u = [-2c, \frac{\sigma^2}{2c})\ ,$$

$$F^u(t) = G_2(t;\frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} >< \frac{\sigma^2, 4c^2}{\sigma^2+4c^2} >)\ .$$

(iii) When $M/2 \le c \le M$,

$$\phi^u = \frac{\sigma^2}{M} - c(1+\frac{\sigma^2}{M^2})\ ,\ \mathbf{x}^u = [-M, \frac{\sigma^2}{M})\ ,$$

$$F^u(t) = G_2(t;\frac{\sigma}{M}) = (< -M, \frac{\sigma^2}{M} >< \frac{\sigma^2, M^2}{\sigma^2+M^2} >)\ .$$

Now, we shall derive the saddle value ϕ^s.
We confine our consideration to the case:

$$0 < c < \sigma^2/M\ . \tag{4.4}$$

On the other hand, it holds that

$$\inf_{F\in\mathcal{F}} \phi(x,F) \le \phi(x, G_2(\cdot; M/\sigma)) = -c \text{ for } x \in [-M, -\sigma^2/M) , \tag{4.2'}$$

$$\inf_{F\in\mathcal{F}} \phi(x,F) \le \phi(x, G_2(\cdot; \sigma/M)) = -\infty \text{ for } x \in [\sigma^2/M, M] , \tag{4.3'}$$

because the player 1 stops immediately in the case of (4.2′) or he cannot stop in the case of (4.3′). Then, the player 1 must decide his stopping level x in the interval

$$\mathbf{x}^M \overset{\text{def}}{=} [-\frac{\sigma^2}{M}, \frac{\sigma^2}{M}) , \tag{4.5}$$

in order not to make his reward $\inf_{F\in\mathcal{F}} \phi(x,F) \le -c$, where $-c$ is the reward of immediately stopping or the saddle value $\phi^s = -c$ in Section 3.

Lemma 7. For any strategy $(x,F), x \in \mathbf{x}^M$, $F \in \mathcal{F}$, if F has a probability mass p at any point y in the interval (x, M) and satisfies $\phi(x,F) \ge -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no mass in the interval (x, M), and it satisfies $\phi(x-0, F'') \le \phi(x,F)$.

Lemma 8. For any strategy $x \in \mathbf{x}^M$, $F \in \mathcal{F}$, if F has probability mass p at any point y in the interval $(-M, x)$ and it satisfies $\phi(x,F) \ge -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no mass in the interval $(-M, x)$, and it satisfies $\phi(x, F'') \le \phi(x,F)$.

Let us define for any $x \in [-\sigma^2/M, \sigma^2/M]$, a three-point cdf $G_3^M(\cdot; x) \in \mathcal{F}$ which has all the mass at three points $-M, x, M$ with the mean 0 and the variance σ^2. This cdf is uniquely determined by

$$G_3^M(t;x) = (< -M, x, M >< \frac{Mx+\sigma^2}{2M(M+x)}, \frac{M^2-\sigma^2}{M^2-x^2}, \frac{\sigma^2 - Mx}{2M(M-x)} >) , \tag{4.11}$$

and let $\mathcal{G}_3^M = \{G_3^M(t;x) \mid -\sigma^2/M \le x \le \sigma^2/M\}$. Note that if $x = \sigma^2/M$ or $-\sigma^2/M$, $G_3^M(t;x)$ becomes the two-point cdf $G_2(t;\sigma/M)$ or $G_2(t; M/\sigma)$ respectively.

The player 1 would decide a stopping level x in the following set

$$\{x \mid \phi(x,F) \ge -c \text{ for all } F \in \mathcal{F}\} \cap \mathbf{x}^M \overset{\text{def}}{=} \mathbf{x}_c^M . \tag{4.13}$$

This set is not empty because $x = -c$ is contained in it.

If there exists a point $x^s \in \mathbf{x}_c^M$ such that

$$\phi(x^s, G_3^M(\cdot; x^s)) = (T^{\ell}_{\mathcal{G}_3^M})^{-1}(c) (= \phi(x^s - 0, G_3^M(\cdot; x^s))) \ge -c , \tag{4.15}$$

the strategy $(x^s, G_3^M(\cdot; x^s))$, $x \in \mathbf{x}_c^M$, $G_3^M(\cdot; x^s) \in \mathcal{G}_3^M \subset \mathcal{F}$, is the saddle point and $\phi^s = (T^{\ell}_{\mathcal{G}_3^M})^{-1}(c)$ is the saddle value. Because, from (4.14), Proposition 1 and (2.10), the following relation is satisfied.

$$\phi(x^s - 0, G_3^M(\cdot; x^s)) \le \sup_{x\in\mathbf{x}_c^M} \inf_{F\in\mathcal{F}} \phi(x,F) \le \inf_{F\in\mathcal{F}} \sup_{x\in\mathbf{x}_c^M} \phi(x,F)$$

$$\leq \inf_{F \in \mathcal{G}_3^M} \sup_{x \in \mathbf{x}_c^M} \phi(x, F) = (T^{\ell}_{\mathcal{G}_3^M})^{-1}(c) \ = \ \phi(x^s, G_3^M(\cdot; x^s)).$$

Theorem 5. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma \leq M$, the saddle point $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ is as follows:
(i) When $\sigma^2/M \leq c \leq M$, the same result as Theorem 3 holds, that is,

$$\phi^s = -c, \ \mathbf{x}^s = [-M, -c] \text{ and}$$

$$\mathcal{F}^s = \{F \mid \int_{[-c,M]} dF(t) = 1, F \in \mathcal{F}(0, \sigma^2, M)\} .$$

(ii) When $0 < c < \sigma^2/M$,

$$\phi^s = (\sigma^2/M - c)^+ - c, \ \mathbf{x}^s = \{x^s\}, \ x^s = (\sigma^2/M - c)^+ - c \text{ and}$$

$$\mathcal{F}^s = \{F^s\}, \ F^s(t) = G_3^M(t; x^s) \text{ defined by (4.11)} .$$

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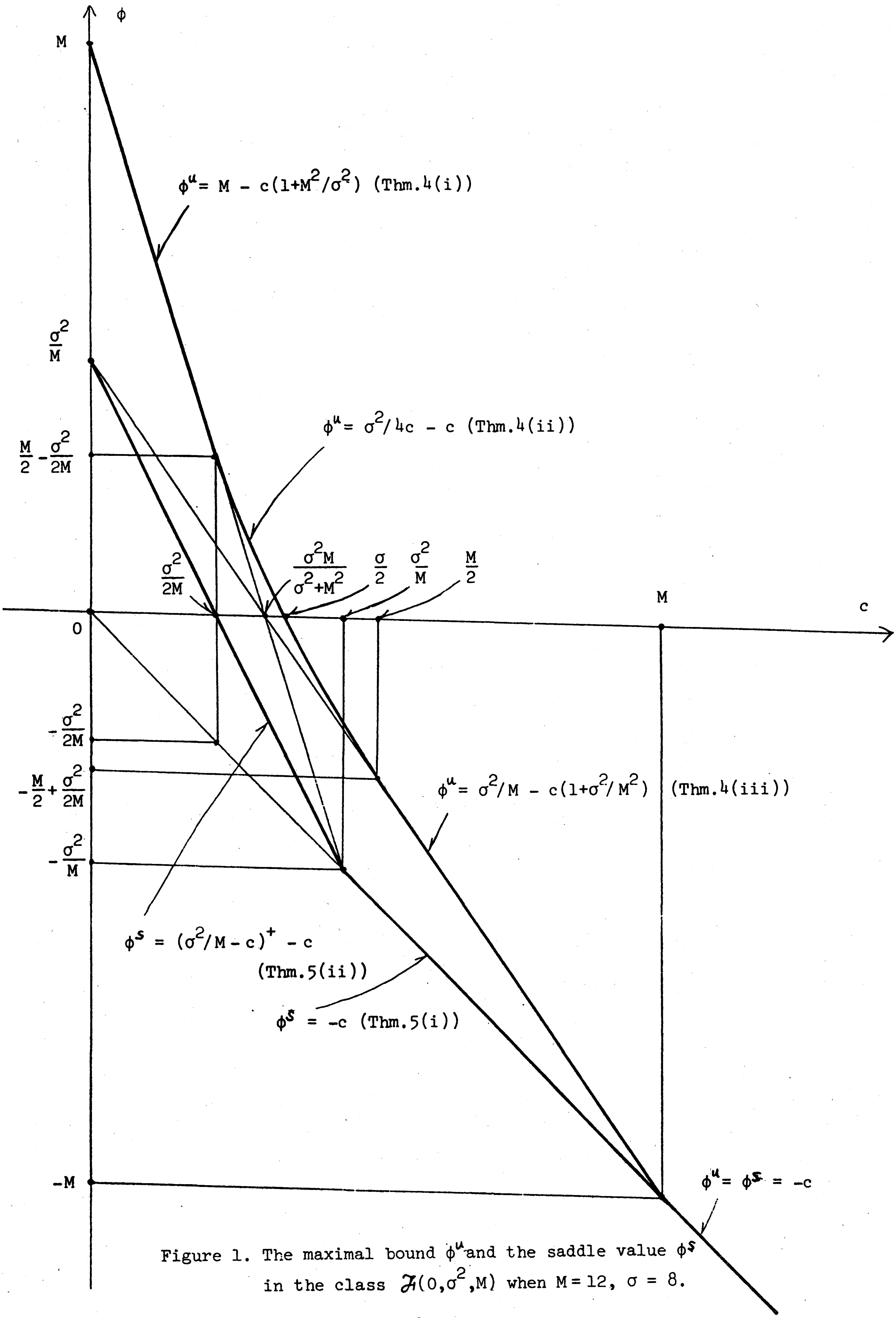

Figure 1. The maximal bound ϕ^u and the saddle value ϕ^s in the class $\mathcal{H}(0,\sigma^2,M)$ when $M = 12$, $\sigma = 8$.